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Semi-parametric estimation for ARCH models

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Abstract In this paper, we conduct semi-parametric estimation for autoregressive conditional heteroscedasticity (ARCH) model with Quasi likelihood (QL) and Asymptotic Quasi-likelihood (AQL) estimation methods. The QL approach relaxes the distributional assumptions of ARCH processes. The AQL technique is obtained from the QL method when the process conditional variance is unknown. We present an application of the methods to a daily exchange rate series.

1. Introduction

The ARCH(q) process is defined by

$$y_t = \mu + \xi_t, \qquad t = 1, 2, 3, \ldots, T. \tag{1.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \cdots + \alpha_q \xi_{t-q}^2 + \zeta_t, \quad t = 1, 2, 3, \ldots, T. \tag{1.2}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. For estimation and applications of (ARCH) models (see, Engle [1,2]; Bollerslev and Kroner [3]; Bera and Higgins [4]; Bollerslev and Nelson [5]; Diebold and Lopez [6]; Pagan [7]; Palm [8]; Shephard [9]; Andersen and Bollerslev [10]; Engle and Patton [11]; Degiannakis and Xekalaki [12]; Diebold [13], and Andersen and Diebold [14]). Moreover, ARCH models have now become standard textbook material in econometrics and finance as exemplified by, e.g., Alexander [15,16], Enders [17], and Taylor [18].

Engle and Gonzalez-Rivera [19] obtained Quasi-maximum-likelihood (QML) estimator to ARCH models that rely on the approximated conditional density by a nonparametric density estimator. Li and Turtle [20] introduced the method of estimating functions to ARCH models. They derived the optimal estimating functions by combining linear and quadratic estimating functions. They also showed that the resultant estimators are more efficient than the QML estimator. Moreover, for semi-parametric and nonparametric estimation of the ARCH models (see, Linton and Mammen [21]; Linton [22]; Su et al. [23]).

Existing techniques for parameter estimation in ARCH models are mainly maximum likelihood based. This means that the probability structure of $\{y_t\}$ has to be known. Usually it assumes $\{y_t\}$ has conditional Gaussian distribution. This concern is very valid in finance as empirical data reveal fat-tails and skewness which contradicts conditional normality. Therefore, estimation procedures may be prone to modeling errors.

This paper applies the Quasi-likelihood (QL) and Asymptotic Quasi-likelihood (AQL) approaches to (ARCH) model.

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The QL approach relaxes the distributional assumptions but has a restriction that assumes the conditional variance process is known. To overcome this limitation, we suggest a substitute technique, the AQL methodology, merging the kernel technique used for parameter estimation of the ARCH model. This AQL methodology enables a substitute technique for parameter estimation when the conditional variance of process is unknown.

This paper is structured as follows. The QL and AQL approaches are introduced and the ARCH model estimation using the QL and AQL methods is developed in Section 2. Reports of simulation outcomes, and numerical cases are presented in Section 3. The QL and AQL techniques are applied to daily exchange rate modeled by ARCH in Section 4. Section 5 summarizes and concludes the paper.

2. Parameter estimation of ARCH(q) model using the QL and AQL methods

In the following, parameter estimation for ARCH(q) model, which includes nonlinear and non-Gaussian models is given. We propose QL and AQL approaches for estimation of ARCH(q) model. The estimations of unknown parameters are considered without any distribution assumptions concerning the processes involved and the estimation is based on different scenarios in which the conditional covariance of the error's terms is assumed to be known or unknown.

2.1. The QL method

Let the observation equation be given by

$$\mathbf{y}_t = \mathbf{f}_t(\boldsymbol{\theta}) + \boldsymbol{\zeta}_t, \quad t = 1, 2, 3, \ldots, T, \tag{2.1.1}$$

$\boldsymbol{\zeta}_t$ is a sequence of martingale difference with respect to $\mathcal{F}_t, \mathcal{F}_t$ denotes the σ-field generated by $\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1$ for $t \geqslant 1$; that is, $E(\boldsymbol{\zeta}_t|\mathcal{F}_{t-1}) = E_{t-1}(\boldsymbol{\zeta}_t) = 0$; where $\mathbf{f}_t(\boldsymbol{\theta})$ is an $\mathcal{F}_{t-1}$ measurable; and $\boldsymbol{\theta}$ is parameter vector, which belongs to an open subset $\Theta \in R^d$. Note that $\boldsymbol{\theta}$ is a parameter of interest. We assume that $E_{t-1}(\boldsymbol{\zeta}_t\boldsymbol{\zeta}'_t) = \boldsymbol{\Sigma}_t$ is known. Now, the linear class $\mathcal{G}_T$ of the estimating function (EF) can be defined by

$$\mathcal{G}_T = \left\{ \sum_{t=1}^{T} \mathbf{W}_t(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta})) \right\}$$

and the quasi-likelihood estimation function (QLEF) can be defined by

$$\mathbf{G}^*_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta})) \tag{2.1.2}$$

where $\mathbf{W}_t$ is $\mathcal{F}_{t-1}$-measureable and $\dot{\mathbf{f}}_t(\boldsymbol{\theta}) = \partial\mathbf{f}_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$. Then, the estimation of $\boldsymbol{\theta}$ by the QL method is the solution of the QL equation $\mathbf{G}^*_T(\boldsymbol{\theta}) = 0$ (see Heyde, [24]).

If the sub-estimating function spaces of $\mathcal{G}_T$ are considered as follows,

$$\mathcal{G}_t = \{\mathbf{W}_t(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta}))\}$$

then the QLEF can be defined by

$$\mathbf{G}^*_{(t)}(\boldsymbol{\theta}) = \dot{\mathbf{f}}_t(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta})) \tag{2.1.3}$$

and the estimation of $\boldsymbol{\theta}$ by the QL method is the solution of the QL equation $\mathbf{G}^*_{(t)}(\boldsymbol{\theta}) = 0$.

A limitation of the QL method is that the nature of $\boldsymbol{\Sigma}_t$ may not be obtainable. A misidentified $\boldsymbol{\Sigma}_t$ could result in a deceptive inference about parameter $\boldsymbol{\theta}$. In the next subsection, we introduce the AQL method, which is basically the QL estimation assuming that the covariance matrix $\boldsymbol{\Sigma}_t$ is unknown.

2.2. The AQL method

The QLEF (see (2.1.2) and (2.1.3)) relies on the information of Σ_t. Such information is not always accessible. To find the QL when $E_{t-1}(\zeta_t\zeta'_t)$ is not accessible, Lin [25] proposed the AQL method.

Definition 2.2.1. *Let $\mathbf{G}^*_{T,n}$ be a sequence of the EF in $\mathcal{G}$. For all $\mathbf{G}_T \in \mathcal{G}$, if*

$$(E\dot{\mathbf{G}}_T)^{-1}(E\mathbf{G}_T\mathbf{G}_T)'(E\dot{\mathbf{G}}'_T)^{-1} - (E\dot{\mathbf{G}}^*_{T,n})^{-1}(E\mathbf{G}^*_{T,n}\mathbf{G}^{*\prime}_T)(E\dot{\mathbf{G}}^{*\prime}_{T,n})^{-1}$$

is asymptotically non-negative definite, $\mathbf{G}^*_{T,n}$ can be denoted as the asymptotic quasi-likelihood estimation function (AQLEF) sequence in $\mathcal{G}$, and the AQL sequence estimates $\boldsymbol{\theta}_{T,n}$ by the AQL method is the solution of the AQL equation $\mathbf{G}^*_{T,n} = 0$.

Suppose, in probability, $\boldsymbol{\Sigma}_{t,n}$ is converging to $E_{t-1}(\boldsymbol{\zeta}_t\boldsymbol{\zeta}'_t)$. Then,

$$\mathbf{G}^*_{T,n}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\boldsymbol{\theta})\boldsymbol{\Sigma}_{t,n}^{-1}(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta})) \tag{2.2.1}$$

expresses an AQLEF sequence. The solution of $\mathbf{G}^*_{T,n}(\boldsymbol{\theta}) = 0$ expresses the AQL sequence estimate $\{\boldsymbol{\theta}^*_{T,n}\}$, which converges to $\boldsymbol{\theta}$ under certain regular conditions.

In this paper, the kernel smoothing estimator of $\boldsymbol{\Sigma}_t$ is suggested to find $\boldsymbol{\Sigma}_{t,n}$ in the AQLEF (2.2.1). A wide-ranging appraisal of the Nadaraya–Watson (NW) estimator-type kernel estimator is available in Härdle [26] and Wand and Jones [27]. By using these kernel estimators, the AQL equation becomes

$$\mathbf{G}^*_{T,n}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\boldsymbol{\theta})\hat{\boldsymbol{\Sigma}}_{t,n}^{-1}(\hat{\boldsymbol{\theta}}^{(0)})(\mathbf{y}_t - \mathbf{f}_t(\boldsymbol{\theta})) = 0. \tag{2.2.2}$$

The estimation of $\boldsymbol{\theta}$ by the AQL method is the solution to (2.2.2). Iterative techniques are suggested to solve the AQL Eq. (2.2.2). Such techniques start with the ordinary least squares (OLS) estimator $\hat{\boldsymbol{\theta}}^{(0)}$ and use $\hat{\boldsymbol{\Sigma}}_{t,n}(\hat{\boldsymbol{\theta}}^{(0)})$ in the AQL Eq. (2.2.2) to obtain the AQL estimator $\hat{\boldsymbol{\theta}}^{(1)}$. Repeat this a few times until it converges.

The next subsections present the parameter estimation of ARCH model using the QL and AQL methods.

2.3. Parameter estimation of ARCH(q) model using the QL method

The ARCH(q) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \ldots, T. \tag{2.3.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \cdots + \alpha_q \xi_{t-q}^2 + \zeta_t, \quad t = 1, 2, 3, \ldots, T. \tag{2.3.2}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. For this scenario, the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_q \xi_{t-q}^2 \end{pmatrix}.$$

The (QLEF), to estimate σ_t^2, is given by

$$G_{(t)}(\sigma_t^2) = (0,1)\begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1}\begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_q \xi_{t-q}^2 \end{pmatrix}$$
$$= \sigma_\zeta^{-2}(\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_q \xi_{t-q}^2). \tag{2.3.3}$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \ldots, \alpha_{q_0}, \sigma_{\zeta_0}^2)$ and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \cdots + \alpha_q \hat{\xi}_{t-q}^2, \quad t = 1, 2, 3, \ldots, T. \tag{2.3.4}$$

The QLEF, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, to estimate the parameters $\mu, \alpha_0, \alpha_1, \ldots, \alpha_q$ is given by

$$G_T(\mu, \alpha_0, \alpha_1, \ldots, \alpha_q) = \sum_{t=1}^{T} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-q}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \alpha_q \xi_{t-q}^2 \end{pmatrix}.$$

The QL estimate of $\mu, \alpha_0, \alpha_1, \ldots, \alpha_q$ is the solution of $G_T(\mu, \alpha_0, \alpha_1, \ldots, \alpha_q) = 0$, where $\hat{\zeta}_t = \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \cdots - \hat{\alpha}_q \hat{\xi}_{t-q}^2, t = 1, 2, 3, \ldots, T$ and

$$\hat{\sigma}_\zeta^2 = \frac{\sum_{t=1}^{T} (\hat{\zeta}_t - \bar{\hat{\zeta}})^2}{T - 1} \tag{2.3.5}$$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\sigma}_\zeta^2)$ is an initial value in the iterative procedure.

2.4. Parameter estimation of ARCH(q) model using the AQL method

For ARCH(q) model given by (2.3.1) and (2.3.2) and using the same argument listed under (2.3.2). Firstly, to estimate σ_t^2, so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0,1)\boldsymbol{\Sigma}_{t,n}^{-1}\begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_q \xi_{t-q}^2 \end{pmatrix}$$

Given $\hat{\xi}_0 = 0, \boldsymbol{\theta}_0 = (\mu_0, \alpha_0, \alpha_1, \ldots, \alpha_q)$, $\boldsymbol{\Sigma}_{t,n}^{(0)} = \mathbf{I}_2$, and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \cdots + \alpha_q \hat{\xi}_{t-q}^2, \quad t = 1, 2, 3, \ldots, T. \tag{2.4.1}$$

Secondly, by kernel estimation method, we find

$$\hat{\boldsymbol{\Sigma}}_{t,n}(\boldsymbol{\theta}^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t, \sigma_t) \\ \hat{\sigma}_n(\sigma_t, y_t) & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Thirdly, to estimate the parameters $\boldsymbol{\theta}_0 = (\mu_0, \alpha_0, \alpha_1, \ldots, \alpha_q)$, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of (AQLEF)

$$G_T(\mu_0, \alpha_0, \alpha_1, \ldots, \alpha_q) = \sum_{t=1}^{T} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-q}^2 \end{pmatrix} \hat{\boldsymbol{\Sigma}}_{t,n}^{-1}$$
$$\times \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_q \xi_{t-q}^2 \end{pmatrix}.$$

The AQL estimate of $\boldsymbol{\theta}_0 = (\mu_0, \alpha_0, \alpha_1, \ldots, \alpha_q)$ is the solution of $G_T(\boldsymbol{\theta}_0) = 0$. The estimation procedure will be iteratively repeated until it converges.

3. Simulation study

For simulation study, We applied QL and AQL approaches for estimation of ARCH(1) model. The ARCH(1) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \ldots, T. \tag{3.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \ldots, T. \tag{3.2}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.

3.1. Parameter estimation of ARCH(1) model using the QL method

For ARCH(1) given by (3.1) and (3.2), the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}.$$

The (QLEF), to estimate σ_t^2, is given by

$$G_{(t)}(\sigma_t^2) = (0,1)\begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1}\begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}$$
$$= \sigma_\zeta^{-2}(\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2). \tag{3.1.1}$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \sigma_{\zeta_0}^2)$ and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2, \quad t = 1, 2, 3, \ldots, T. \tag{3.1.2}$$

To estimate the parameters μ, α_0 and α_1 , using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, The QLEF is given by

$$G_T(\mu, \alpha_0, \alpha_1) = \sum_{t=1}^{T} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}.$$

The solution of $G_T(\mu,\alpha_0,\alpha_1)=0$ is the QL estimate of μ,α_0 and α_1. Therefore

$$\hat{\mu}=\sum_{t=1}^{T}\frac{y_t}{\hat{\sigma}_t^2}\Big/\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_t^2}. \tag{3.1.3}$$

$$\hat{\alpha}_1=\frac{T\sum_{t=1}^{T}\hat{\sigma}_t^2\hat{\xi}_{t-1}^2-\sum_{t=1}^{T}\hat{\sigma}_t^2\sum_{t=1}^{T}\hat{\xi}_{t-1}^2}{T\sum_{t=1}^{T}\hat{\xi}_{t-1}^4-\left(\sum_{t=1}^{T}\hat{\xi}_{t-1}^2\right)^2}. \tag{3.1.4}$$

$$\hat{\alpha}_0=\frac{\sum_{t=1}^{T}\hat{\sigma}_t^2-\hat{\alpha}_1\sum_{t=1}^{T}\hat{\xi}_{t-1}^2}{T}. \tag{3.1.5}$$

and let

$$\hat{\sigma}_\zeta^2=\frac{\sum_{t=1}^{T}(\hat{\zeta}_t-\bar{\hat{\zeta}})^2}{T-1} \tag{3.1.6}$$

where $\hat{\zeta}_t=\hat{\sigma}_t^2-\hat{\alpha}_0-\hat{\alpha}_1\hat{\xi}_{t-1}^2, t=1,2,3,\ldots,T$.

$\hat{\psi}=\left(\hat{\mu},\hat{\alpha}_0,\hat{\alpha}_1,\hat{\sigma}_\zeta^2\right)$ is an initial value in the iterative procedure.

The initial values might be affected by the estimation results. For extensive discussion on assigning initial values in the (QL) estimation procedures (see Alzghool and Lin [28,29]).

3.2. Parameter estimation of ARCH(1) model using the AQL method

For ARCH(1) model given by (3.1) and (3.2) and using the same argument listed under (3.1.2). Firstly, to estimate σ_t^2, so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2)=(0,1)\mathbf{\Sigma}_{t,n}^{-1}\begin{pmatrix} y_t-\mu \\ \sigma_t^2-\alpha_0-\alpha_1\xi_{t-1}^2\end{pmatrix}$$

Given $\hat{\xi}_0=0, \boldsymbol{\theta}_0=(\mu_0,\alpha_0,\alpha_1,\mu_0)$, $\mathbf{\Sigma}_{t,n}^{(0)}=\mathbf{I}_2$ and $\hat{\xi}_{t-1}^2=(y_{t-1}-\mu_0)^2$, then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2)=0$, that is,

$$\hat{\sigma}_t^2=\alpha_0+\alpha_1\hat{\xi}_{t-1}^2, \quad t=1,2,3,\ldots,T. \tag{3.2.1}$$

Secondly, by kernel estimation method, we find

$$\hat{\mathbf{\Sigma}}_{t,n}(\boldsymbol{\theta}^{(0)})=\begin{pmatrix}\hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t,\sigma_t)\\ \hat{\sigma}_n(\sigma_t,y_t) & \hat{\sigma}_n(\sigma_t)\end{pmatrix}.$$

Thirdly, to estimate the parameters $\boldsymbol{\theta}=(\mu,\alpha_0,\alpha_1)$, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of (AQLEF)

$$G_T(\mu,\alpha_0,\alpha_1)=\sum_{t=1}^{T}\begin{pmatrix}-1 & 0\\ 0 & -1\\ 0 & -\hat{\xi}_{t-1}\end{pmatrix}\hat{\mathbf{\Sigma}}_{t,n}^{-1}\begin{pmatrix} y_t-\mu\\ \sigma_t^2-\alpha_0-\alpha_1\xi_{t-1}^2\end{pmatrix}.$$

The AQL estimate of γ,ϕ, and μ is the solution of $G_T(\mu,\alpha_0,\alpha_1)=0$. Therefore

$$\hat{\mu}=\sum_{t=1}^{T}\frac{y_t}{\hat{\sigma}_n(y_t)}\Big/\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(y_t)}. \tag{3.2.2}$$

$$\hat{\alpha}_1=\frac{\left(\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)}\right)\left(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}\right)-\left(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)}\right)\left(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}\right)}{\left(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}\right)^2-\left(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)}\right)\left(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)}\right)}. \tag{3.2.3}$$

$$\hat{\alpha}_0=\frac{\left(\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)}\right)-\hat{\alpha}_1\left(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}\right)}{\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)}}. \tag{3.2.4}$$

and let

$$\hat{\sigma}_\zeta^2=\frac{\sum_{t=1}^{T}(\hat{\zeta}_t-\bar{\hat{\zeta}})^2}{T-1} \tag{3.2.5}$$

Table 1 The QL and AQL estimates and The RMSE of each estimates is stated below that estimate.

	α_0	α_1	μ	α_0	α_1	μ	α_0	α_1	μ
True	0.010	0.980	1.30	0.010	0.980	−1.30	0.010	0.980	0.030
QL	0.009	0.989	1.299	0.009	0.989	−1.30	0.009	0.989	0.029
	0.001	0.010	0.006	0.001	0.010	0.006	0.001	0.010	0.006
AQL	0.009	0.989	1.30	0.009	0.989	−1.29	0.009	0.989	0.030
	0.001	0.010	0.0003	0.002	0.009	0.0003	0.001	0.009	0.0003
True	0.050	0.950	1.30	0.050	0.950	−1.30	0.050	.950	0.030
QL	0.049	0.949	1.29	0.049	0.940	−1.30	0.049	0.94	0.029
	0.001	.0001	0.014	0.001	0.010	0.014	0.001	0.010	0.014
AQL	0.049	0.940	1.32	0.049	0.940	−1.30	0.049	0.940	0.032
	0.001	0.010	0.018	0.001	0.010	0.018	0.001	0.01	0.001
True	0.10	0.90	1.30	0.10	0.90	−1.30	0.10	.90	0.030
QL	0.098	0.910	1.29	0.098	0.910	−1.30	0.098	0.910	0.023
	0.002	0.010	0.019	0.002	0.010	0.020	0.002	0.010	0.029
AQL	0.098	0.910	1.31	0.098	0.910	−1.32	0.098	0.910	0.031
	0.002	0.010	0.012	0.002	0.010	0.021	0.001	0.010	0.001
True	0.1	0.90	−0.03	0.05	0.95	−0.03	0.01	.98	−0.03
QL	0.098	0.910	−0.031	0.051	0.949	−0.030	0.009	0.990	−0.030
	0.002	0.010	0.019	0.001	0.001	0.014	0.001	0.016	0.006
AQL	0.098	0.910	−0.031	0.051	0.949	−0.031	0.009	0.990	−0.031
	0.002	0.010	0.001	0.001	0.001	0.002	001	0.010	0.001

Table 2 The QL and AQL estimates and the RMSE of each estimates is stated below that estimate.

		α_0	α_1	μ	α_0	α_1	μ
$T = 20$	True	0.010	0.980	−0.030	0.05	0.950	1.3
	QL	0.009	0.990	−0.029	0.0495	0.9485	1.300
		0.0008	0.0100	0.0319	0.0005	0.0015	0.0703
	AQL	0.009	0.990	−0.031	0.0495	0.9485	1.3107
		0.0009	0.010	0.0084	0.0005	0.0015	0.0213
$T = 40$	QL	0.009	0.990	−0.031	0.0495	0.9485	1.3015
		0.00089	0.010	0.0223	0.0005	0.0015	0.0492
	AQL	0.009	0.990	−0.031	0.0495	0.9485	1.3113
		0.00089	0.010	0.0039	0.0005	0.0015	0.0143
$T = 60$	QL	0.009	0.990	−0.029	0.0495	0.9485	1.300
		0.0009	0.010	0.0180	0.0005	0.0015	0.0404
	AQL	0.009	0.990	−0.031	0.0495	0.9485	1.311
		0.0009	0.010	0.0027	0.0005	0.0015	0.0128
$T = 80$	QL	0.009	0.990	−0.029	0.0490	0.9485	1.300
		0.0009	0.010	0.016	0.0005	0.0015	0.0353
	AQL	0.009	0.990	−0.310	0.0495	0.9485	1.3112
		0.0009	0.010	0.0020	0.0005	0.0015	0.0119
$T = 100$	QL	0.009	0.990	0.0292	0.0495	0.9485	1.3017
		0.0009	0.010	0.0142	0.0005	0.0015	0.0314
	AQL	0.009	0.990	−0.031	0.0495	0.9485	1.3111
		0.0009	0.010	0.0018	0.0005	0.0015	0.0116

The estimation procedure will be iteratively repeated until it converges.

For this simulation study, samples of size $T = 500$ are taken, and the mean and root mean squared errors (RMSE) for $\hat{\alpha}_0, \hat{\alpha}_1$, and $\hat{\mu}$ are calculated, where $N = 1000$ independent samples. In Table 1, QL represents the QL estimate and AQL represents the AQL estimate.

The effect of the sample size on the estimation of parameters is considered. Samples of sizes $T = 20$, 40, 60, 80, and 100 were generated. In Table 2, The results are revealed that the RMSE will be decreased when the sample size is increased.

4. Application to ARCH model

The QL and AQL methods developed in earlier section apply to real data where the data are modeled by ARCH model (1.1) and (1.2). The data set contains the daily exchange rate of

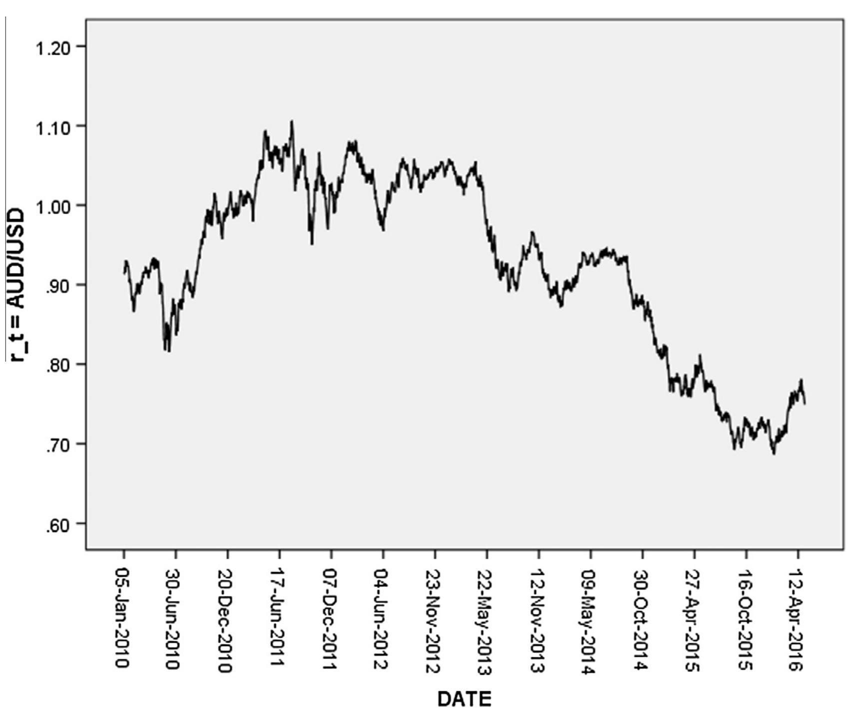

Figure 4.1 The plot of the daily exchange rates of $r_t = AUD/USD$ (Australian Dollar / US Dollar).

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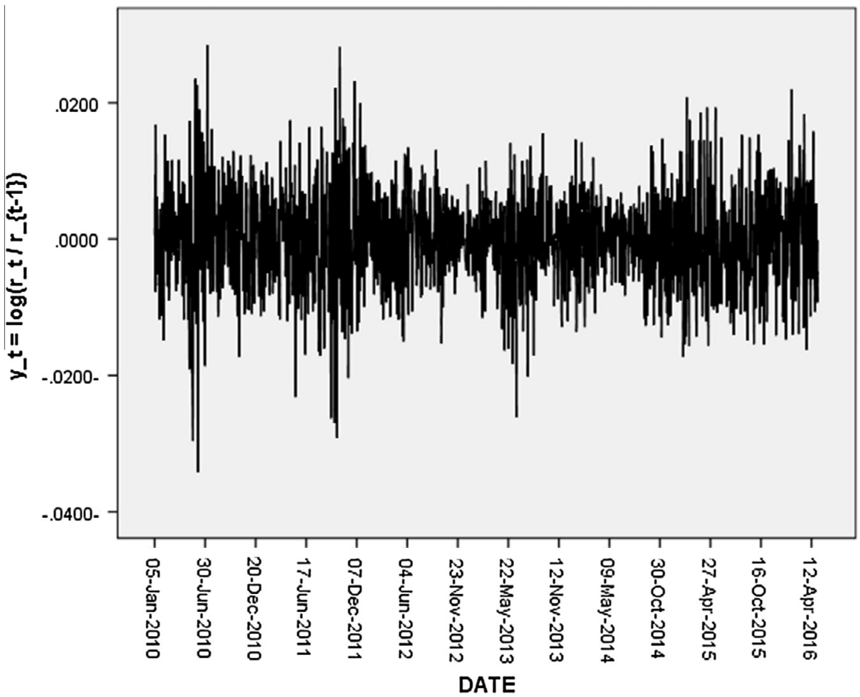

Figure 4.2 The plot of $y_t = \log(r_t/r_{t-1})$.

Table 3 Estimation of α_0, α_1, μ for the exchange rate Pound/Dollar data.

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\mu}$	$\frac{\bar{\hat{\xi}}_t}{S.d(\hat{\xi}_t)}$
QL	0.1300	0.8387	−0.00012	0.00013
AQL	0.0200	0.9599	−0.00111	0.1350

$r_t = AUD/USD$ (Australian Dollar/US Dollar) for period from 5/6/2010 to 5/5/2016, 1590 observations in total (see http://www.rba.gov.au/statistics/historical-data.html). r_t appear not to be stationary, as indicated in Fig. 4.1. We took the nature logarithm of r_t, and let $y_t = \log(r_t/r_{t-1})$, $t = 1, 2, 3, \ldots, 1590$. The series of y_t is presented in Fig. 4.2.

We used the S + FinMetrics function archTest to carry out Lagrange Multiplier (ML) test for the presence of ARCH effects in the residuals (see; Zivot and Wang [30]). For r_t the p-value is significant (< 0.05 level), so reject the null hypothesis that there are no ARCH effects and we fit $\{y_t\}$ by following models:

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \ldots, T. \tag{4.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \ldots, T. \tag{4.2}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.

Table 3 presents the estimates of α_0, α_1, and μ achieves by two methods. QL represents the estimate found by QL method, and AQL represents the asymptotic quasi-likelihood estimate.

We can see from the fourth column in Table 3 that QL gives smaller standardized residuals. The QL method tends to be more efficient than AQL method.

5. Summary

In this paper, the estimation of the parameters in ARCH models has been presented by two alternative approaches. The article has shown that the QL and AQL estimating procedures provide an efficient approach for estimating the unknown parameter when the exactly probability structure of underlying model is unknown. It will provide a robust tool for obtaining optimal point estimate of parameters in heteroscedastic models, like ARCH model.

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